Revue des Energies Renouvelables Vol. 22 N°2 (2019) 251 - 256
DOI: https://doi.org/10.54966/jreen.v22i2.742

Potential flow over an inclined thin flat-plate

Chawki Mahboub *, Noureddine Moummi and Abdelhafid Brima

Laboratory of Mechanical Engineering
University of Biskra, Mohamed Khider
B.P. 145, R.P. 07000 Biskra, Algeria

(reçu le 21 Juin 2019 - accepté le 28 Juin 2019)

Résumé - *Ce travail vise à prédire avec précision la distribution de la vitesse sur le côté d'une plaque plane face à un écoulement uniforme avec une inclinaison arbitraire. L'équation de Laplace régissant le problème a été résolue avec la méthode des différences finies et les résultats ont été comparés avec la théorie. Une corrélation simple de la distribution de vitesse est établie.*

Abstract - *This work aims to accurately predict the velocity distribution along a flat-plate surface facing a uniform flow with arbitrary inclination. The Laplace equation governing the problem was solved with the finite difference method, and the results were compared with those come from theory. A simple correlation of the velocity distribution is established.*

Mots-clés: Ecoulement non-visqueux irrotationnel - Plaque plane inclinée - Théorie des écoulements potentiels - Point de stagnation.

1. INTRODUCTION

Prandtl's boundary layer theory implies that in flows at high Reynolds numbers the velocity in the boundary layer varies quickly from zero directly on a bounding surface up to a finite value which corresponds to the inviscid limiting solution $Re \to \infty$ [1].

Therefore, the velocity field of the inviscid flow region, where the viscous effects can be neglected, needs to be first determined before one can proceed to obtain the velocity distribution in the boundary layer, where the viscous effects are significant.

The external flow over a flat-plate is a very common configuration in many engineering applications such as airfoils [2], solar collectors and building roofs [3], fibrous filters [4], etc. The inviscid velocity field around a flat-plate can be obtained using the two-dimensional potential flow approximation.

The expression of the local velocity component parallel to the plate surface (y = 0) facing the uniform incoming flow can be derived from the conformal transformation of the potential flow solution past a circular cylinder or from Hess' theory as:

$$\frac{u}{U_\infty} = \cos\alpha + f\left(x/a\right)\sin\alpha \tag{1}$$

With f (x/a) the dimensionless distribution of the velocity for flow normal to the flat-plate ($\alpha = \pi/2$) is given by:

$$f(x/a) = \pm\left(\frac{(x/a)^2}{1-(x/a)^2}\right) \qquad \text{(conformal transformation)} \tag{2}$$

or

* c.mahboub@univ-biskra.dz - n.moummi@univ-biskra.dz
- a.brima@univ-biskra.dz

$$f(x/a) = \frac{1}{\pi}\ln\left(\frac{1+(x/a)}{1-(x/a)}\right) \qquad \text{(Hess' theory)} \tag{3}$$

where x and y are the axes respectively parallel and normal to the flat-plate. Despite the different results produced by (2) and (3), we can clearly see that both relations give an infinite velocity at the edges of the plate, which does not allow eventually for the determination of the boundary layer solution.

Accordingly, this study was conducted in order to investigate the present problem and to provide a bounded distribution of the potential velocity along the flat-pate surface, as well as to locate the stagnation point.

2. FORMULATION AND FINITE DIFFERENCE SOLUTION

The flat plate of length 2a is inclined to the horizontal free stream at an angle α varying from 0 to $\pi/2$ as shown in figure 1. The flow is considered to be inviscid, incompressible and irrotational, so that the stream function satisfies the Laplace equation which is written in polar coordinates as:

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0 \tag{4}$$

which has to be solved subject to the following boundary conditions,

$$\begin{array}{llll} \Psi = 0 & \text{on} & \theta = \alpha & 0 \le r \le a \\ \Psi = 0 & \text{on} & \theta = \alpha + \pi & 0 \le r \le a \\ \Psi = U_\infty r \sin 0 & \text{as} & r \to \infty & 0 \le \theta \le 2\pi \end{array} \tag{5}$$

To obtain the numerical solution of the equation (4) in a domain of infinite extent, it is more convenient to introduce the following dimensionless transformation

$$\xi = 0 \qquad \eta = \frac{a}{r+a} \qquad \psi = \frac{\Psi}{U_\infty a} \tag{6}$$

Then the equation (4) and the boundary conditions (5) become,

$$\eta^2 (1-\eta)^2 \frac{\partial^2 \Psi}{\partial \eta^2} + \eta(1-\eta)(1-2\eta)\frac{\partial \Psi}{\partial \eta} + \frac{\partial^2 \Psi}{\partial \xi^2} = 0 \tag{7}$$

$$\begin{array}{llll} \Psi = 0 & \text{on} & \xi = \alpha & 1/2 \le \eta \le 1 \\ \Psi = 0 & \text{on} & \partial = \alpha + \pi & 1/2 \le \eta \le 1 \\ \Psi = \dfrac{1-\eta}{\eta}\sin\xi & \text{as} & \eta \to 0\infty & 0 \le \xi \le 2\pi \end{array} \tag{8}$$

Using a central finite difference scheme, the transformed equation (7) is rearranged as:

$$\begin{aligned} 2\left(\left(\frac{\Delta\xi}{\Delta\eta}\eta_j(1-\eta_j)\right)^2 + 1\right)\Psi_{i,j} &= \Psi_{i+1,j} + \Psi_{i-1,j} + \\ &\left(\left(\frac{\Delta\xi}{\Delta\eta}\eta_j(1-\eta_j)\right)^2 + \frac{(\Delta\xi)^2}{2\Delta\eta}\eta_j(1-\eta_j)(1-2\eta_j)\right)\Psi_{i,j+1} + \\ &\left(\left(\frac{\Delta\xi}{\Delta\eta}\eta_j(1-\eta_j)\right)^2 + \frac{(\Delta\xi)^2}{2\Delta\eta}\eta_j(1-\eta_j)(1-2\eta_j)\right) + \Psi_{i,j-1} \end{aligned} \tag{9}$$

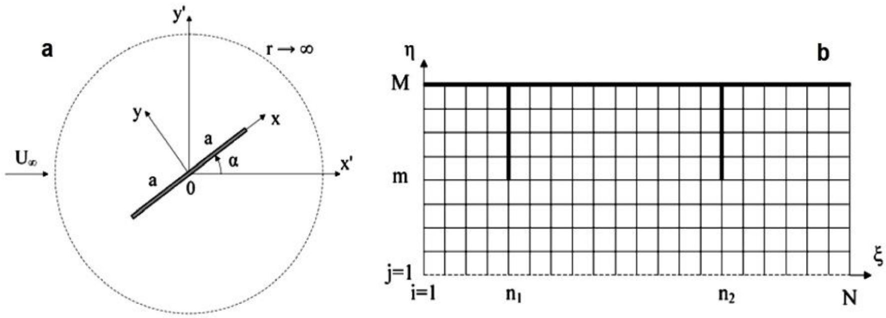

Fig. 1: **(a)** Physical domain, **(b)** Computational domain

and the boundary conditions (8) read as:

$$\begin{aligned} &\Psi_{i,M} = 0 \quad \text{and} \quad \Psi_{i,1} = \frac{1-n_0}{n_0}\sin\xi_i && \text{for} \quad i = 1 \to N \\ &\Psi_{1,i} = \Psi_{N,j} && \text{for} \quad j = 1 \to M \\ &\Psi_{n_1,i} = \Psi_{n_2,j} = 0 && \text{for} \quad j = m \to M \end{aligned} \tag{10}$$

where the flat plate surface is indicated in the computational domain by thick lines correspond to grid points identified by ($i = n_1$, $j = m \to M$) , ($i = n_2$, $j = m \to M$) and ($i = 1 \to N$, $j = M$). The point successive over-relaxation method is used to solve the equation (9) and the criterion, $\max\left|\Psi^{k}_{i,j} - \Psi^{k-1}_{i,j}\right| / \left|\Psi^{k}_{i,j}\right| < 10^{-10}$, is used at the end of each iteration to determine the convergence of the procedure.

The computations were performed for $\eta_0 = 0.1$ (which corresponds to $r_\infty = 9a$) and for different grids of N×M points in the ξ- and η-direction respectively with uniform grid spacing $\Delta\xi$ and $\Delta\eta$ until results become less sensitive to the grid refinement.

Once the stream function is obtained the velocity can be determined from the Cauchy-Riemann equations. Since at the surface of the flat plate we have, $x \equiv \pm r$ and $u \equiv \pm V_r$, the velocity along the flat plate surface facing the uniform incoming flow can be calculated by:

$$\frac{u}{U_\infty} = \pm\frac{\eta}{1-\eta}\frac{\partial\Psi}{\partial\xi} \quad \text{and} \quad \frac{x}{a} = \pm\frac{\eta}{1-\eta} \tag{11}$$

where the minus (plus) sign corresponds to the first (second) half length of the plate. In finite difference approximation, the equation (11) can be written as:

$$\frac{u}{U_\infty} = \begin{cases} \dfrac{\eta_j}{1-\eta_j}\dfrac{\Psi_{i+1,j} - \Psi_{i,j}}{\Delta\xi} & \text{for} \quad \dfrac{x_j}{a} = \dfrac{\eta_j}{1-\eta_j} \\ -\dfrac{\eta_j}{1-\eta_j}\dfrac{\Psi_{i,j} - \Psi_{i-1,j}}{\Delta\xi} & \text{for} \quad \dfrac{x_j}{a} = \dfrac{\eta_j}{1-\eta_j} \end{cases} \tag{12}$$

The choice of the forward and backward difference schemes in (12) is not arbitrary but rather dictated by the flat plate surface - facing the uniform incoming flow - over which we need to determine the velocity distribution.

3. RESULTS

The numerical results show that $f(x/a) = \left(\frac{u}{U_a} - \cos\alpha\right) \Big/ \sin\alpha$ is independent of the angle of inclination which means that the equation (1) holds; however, neither equation (2) nor equation (3) can successfully predict $f(x/a)$ over the whole flat-plate surface.

The figure 2 shows that the dimensionless velocity increases almost linearly for 0 < x/a < 0.6, then it increases more rapidly up to a maximum value of about 6.65 at the edge of the plate.

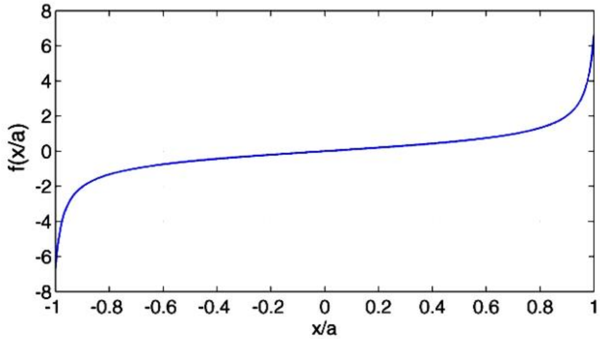

Fig. 2: Variation of the dimensionless velocity along the flat plate surface for an inclination angle of 90°

The comparison of the numerical results and the analytical solutions in the **Table 1** shows that the equation comes from the conformal transformation agrees very well with the numerical results on the greater portion of the surface of the plate except near the edges; whereas the equation comes from Hess' theory gives inaccurate predictions over the entire length of the plate.

Based on the results of the present analysis and using OriginPro software, the function $f(x/a)$ can be represented by the following approximate expression with relative error less than 5 %,

$$f(x/a) = \frac{1.03183\ x/a)}{\cos(\arcsin(0.98891\ x/a))} \quad (13)$$

The position of the stagnation point x_{sp} is also of interest when treating, for example, the heat transfer problem of such configuration under the assumptions of the boundary layer theory [3].

It can be derived as a function of the angle of inclination by setting x = 0 in the equation (1), which leads to the following expressions,

$$\frac{x_{sp}}{a} = -\left(\frac{(\cot\alpha)^2}{1+(\cot\alpha)^2}\right)^{1/2} \quad \text{(confor mal transformation)} \quad (14)$$

or

$$\frac{x_{sp}}{a} = -\frac{1 - \exp(-\pi\cot\alpha)}{1 + \exp(-\pi\cot\alpha)} \quad \text{(Hess' theory)} \quad (15)$$

The position of the stagnation point calculated from the equations (14) and (15), and obtained from the numerical solution at different angles of inclination is shown in the Table 2.

The results show a very good agreement between the numerical solution and the equation (14) for the entire range of the angle of inclination; whereas we can see that the numerical solution does not agree with the equation (15).

Table 1. Comparison of the numerical solution and analytical solutions of f (x/a)

η	Numerical solution	Eq. (2)	Eq. (3)
0.5	6.6487	∞	∞
0.6	0.8927	0.8944	0.5123
0.7	0.4747	0.4743	0.2917
0.8	0.2586	0.2582	0.1626
0.9	0.1120	0.1118	0.0710
1.0	0.0000	0.0000	0.0000

Table 2. Comparison of the numerical solution and analytical solutions of (x_{sp} / a)

α (°)	Numerical solution	Eq. (14)	Eq. (15)
15	-0.9714	-0.9659	-1.0000
30	-0.8688	-0.8660	-0.9914
45	-0.7081	-0.7071	-0.9172
60	-0.4999	-0.5000	-0.7196
75	-0.2585	-0.2588	-0.3977
90	0.0000	0.0000	0.0000

4. CONCLUSION

From the results presented above, in the boundary layer type flow occurring along a flat-plate that makes an angle α with the free stream velocity U_∞, the position of the stagnation point x_{sp} is given by the equation (14), while the velocity distribution in the outer flow region can be predicted accurately using the equation (1) combined with the equation (13).

NOMENCLATURE

a, Half length of the plate	α, Plate inclination
r, θ, Polar coordinates	ξ, η, Dimensionless coordinates
u, Velocity parallel, flat-plate surface.	ψ, Stream function
U_∞, Velocity of the uniform flow	Ψ, Dimensionless stream function

REFERENCES

[1] H. Schlichting and K. Gersten, '<u>*Boundary-Layer Theory*</u>', 9th Ed, Springer-Verlag: Berlin, Germany, 2017.

[2] Q. Sun and I.D. Boyd, '*Flat-plate aerodynamics at very low Reynolds number*', Journal of Fluid Mechanics, Vol. 502, pp. 199 - 206, 2004.

[3] A.A. Kendoush, '*Theoretical analysis of heat and mass transfer to fluids flowing across a flat plate*', International of Thermal Sciences, Vol. 48, pp. 188 - 194.

[4] M. Xie, Q. He and W. Wang, '*Interception Efficiency of Particle Laden Flow over a Finite Flat Plate in Potential Flow Regime*s', International Thermal Science, Vol. 17, pp. 1343 - 1348, 2013.

[5] J. Katz, and A. Plotkin, '*Low-Speed Aerodynamics*', 2nd ed., Cambridge University Press, New York, USA, 2001.

[6] J.L. Hess, '*Analytic solutions for potential flow over a class of semi-infinite two-dimensional bodies having circular-arc noses*', Journal of Fluid Mechanics, Vol. 60, pp. 225 - 239, 1973.